Integrals of composite functions through tabular integration by parts

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ABSTRACT

The paper aims to expose the applications of Tabular Integration by Parts (TIBP) in evaluating the integrals of composite functions. In the study, the algorithm of the tabular integration by parts was used in integrating of the products of the two elementary functions resulting from composition conversion of the forms $\int f(x^{1/n})dx$ *and* $\int f(\ln x)dx$. *Examples are also analyzed to identify other forms of composite functions where the TIBP can be applied.*

Keywords
Elementary Functions, Composite Functions, Integrals, Tabular Integration by Parts (TIBP)

Introduction

Integration by parts is a technique for simplifying integrals of the form $\int f(x)\, g(x)dx$ in which *f(x)* can be differentiated repeatedly and *g(x)* integrated repeatedly without difficulty. It introduces us to a technique of great adaptability and wide application. As a method within a major branch of calculus, integration-by-parts by itself can provide more than one road into that precise answer. In most of the calculus books, the integration by parts formula is expressed as $\int udv = uv - \int vdu$.

An acronym that is very helpful to remember when using integration by parts is LIATE. The rule was first mentioned in a paper of H. E. Kasube (1983) [6]. The word LIATE stands for Logarithmic, Inverse trigonometric, Algebraic, Trigonometric, Exponential. Based on the rule, if there is a combination of two of these types of functions in the original integral, choose for "u" the type that appears first in the LIATE and "dv" is whatever is left, i.e, the part that appears second in the LIATE. In particular, whichever function comes first in the following list should be "u"

L (Logarithmic): $\ln x$, $\log_b x$, etc.,
I (Inverse trigonometric): arctan x, arcsec x,
A (Algebraic): x^2, $3x^{50}$, etc.,
T (Trigonometric): sin x, tan x, etc.,
E (Exponential): e^x, 19^x, etc.

Basically, the function which is to be "dv" comes later in the list, *i.e*, functions lower on the list have easier anti-derivatives than the functions above them. Students have liked using LIATE and found choosing "u" and "dv" much less frightening.. In another paper, Nicol, S.J.[9] commented that nearly every calculus text that has been encountered in the past several years uses the identities to evaluate integrals of certain form. Most students balk in anticipation of more formulas to memorize. These integrals are typically found in the section of a text dealing with integrating powers of trigonometric functions, which follows the section on integration by parts. It can be contended that these integrals should be done by repeated (iterated) integration by parts. Finally, for the student who has been taught tabular integration by parts the calculation will consider the integral that is evaluated without the use of trigonometric identities and, preferably, in terms of the arguments of the trigonometric functions found in the original problem. Incidentally, checking the example presented and a few others by differentiation may prompt some to notice the forms that appear as antiderivatives and thereby to sense the possibility of yet another method: the undetermined coefficients.

Tabular integration by parts was featured in the article of Gillman, L. [4] which commented on the engaging article by David Horowitz [5]. The method is based on iterating the diagram and can be ended at any stage. Tabular integration by parts streamlines the integrations and also makes proofs of operational properties more elegant and accessible.

u ———— *v'* The integral of this product

equals this product

u' ———— *v* minus the integral of this product,

The notes of Kowalski [13] commented that the integration by parts formula can be difficult to apply repeatedly; it takes a lot of space to write down and facilitates making distribution error. Fortunately, there is a purely mechanical procedure for performing integration by parts without writing down the so much called Tabular Integration by Parts. The notes presented theorem and tables to illustrate the process.

In an article by Switkes, J. S. [12], the technique of Integration by Parts as an integration rule corresponding to the Product Rule for differentiation was introduced. The derivation of a Quotient Rule Integration by Parts formula was presented, applying the resulting integration formula to an example, and discuss reasons why this formula does not appear in calculus texts.

Column Integration or tabular integration by parts was cited in an article of Dence, T. P. [2] as a method that calls for two columns of functions, labeled D and I. To determine the integral of the form $\int f(x)g'(x)dx$, *f(x)* is placed under D column, then successively differentiates it, while *g'(x)* is placed under I column followed by successive antiderivatives. Then, in many cases, a series of cross-multiplications, with alternating signs converges to the integral $\int f(x)g'(x)dx$. In most applications *f(x)* is a polynomial, so the D column ends after finitely many steps and column integration yields$\int f(x)g'(x)dx$. If *f(x)* is not a polynomial, one can always truncate the process at any level and obtain a remainder term defined as the integral of the product of the two terms directly across from each other, namely $f^{(k)}(x)$ and $g^{(-k+1)}(x)$. Then, the infinite series would converge if the remainder tends to zero.

This integration by parts is a technique for evaluating an integral of a product of functions taught to college students taking Calculus-2. The goal of the integration by parts is to go from an integral $\int u\,dv$ that one finds it difficult to evaluate to an integral $\int v\,du$ that can be easily evaluated. Some integrals require repeated applications of integration by parts, which can become tedious and troubling for students, so the tabular method was developed from the integration by parts technique. It shortcuts the process and organizes work in table. Unfortunately this method is not commonly taught or utilized. Many popular calculus textbooks do not even mention the tabular method and, if they do, it is usually brief. One of the explorations conducted by the researcher is presented in the paper [13] that exposed how the modified algorithm of the Tabular Integration by Parts (TIBP) was used to derive some reduction formula and to prove some theorems in mathematical analysis. The present paper exposed the applications of the TIBP in evaluating the integrals of the products of functions resulting from composition conversion. Obtaining the composite function of two given functions is another operation on functions. Given the two functions *f* and *g*, the composite function [8], denoted by $f \circ g$, is defined by $(f \circ g)(x) = f[g(x)]$ and the domain of $f \circ g$ is the set of all numbers *x* in the domain of *g* such that *g(x)* is in the domain of *f*. The definition indicates that when computing $(f \circ g)(x)$, the function *g* is applied to *x* first and then function *f* to *g(x)*.

The general objective of the paper is to expose the applications of Tabular Integration by Parts (TIBP) in evaluating the integrals of composite functions. Specifically, it aims to

1. Illustrate the use of the algorithm of the Tabular Integration by Parts.

2. Determine the general forms of composite functions where TIBP can be applied.

3. Evaluate the integrals of composite functions using TIBP.

The study is very significant to the students to gain further insights into the use of Tabular Integration by Parts as an alternative, efficient and elegant way of finding the integrals of product of elementary functions are provided. It aims to help the readers appreciate the techniques of integration in a more interesting way.

Methods

This study as a pure research is both descriptive and expository in nature and like any other researches of this type, it would follow its own form and style. As a descriptive research, this would more than describe the concepts in mathematics, as it exposes and clarifies the formulae and propositions fitted to the study.

Results and Discussion

Algorithm of the Tabular Integration by Parts

The algorithm of the Tabular Integration By Parts (TIBP) is based on the algorithm given by Horowitz [7]. To illustrate, the table below presents the algorithm:

Table 1
Algorithm of TIBP for evaluating $\int f(x)g(x)dx$.

S	D	I
+ →	$f(x)$	$g(x)$
- →	$f^1(x)$	$g^{[1]}(x)$
+ →	$f^2(x)$	$g^{[2]}(x)$
- →	$f^3(x)$	$g^{[3]}(x)$
$\vdots$	$\vdots$	$\vdots$
$(-1)^n$ →	$f^n(x)$	$g^{[n]}(x)$
$(-1)^{n+1}$ →	$f^{n+1}(x)$ →	$g^{[n+1]}(x)$

In column 1 (the S Column) of the table write alternating plus and minus signs. In column 2 (the D column), list f(x) and its successive derivatives. In column 3 (the I column) list g(x) and its successive antiderivatives. Form successive terms by multiplying each entry in the S and D columns by the entry in the find column that lies ***below*** it. The resulting sum of these terms is the integral. If f(x) is a polynomial, then there will be only a finite number of terms to add. Otherwise, the process may be truncated at any level by forming a remainder term defined as the product of the integral of the entries in S and D columns and the entry in the first column that lies directly ***across*** from it. In this table, $f^i(x)$ denotes the *i^th^* derivative of the function $f(x)$ while $g^{[i]}(x)$ denotes the *i^th^* integration (antiderivative) of $g(x)$.

The following examples illustrate the use of the algorithm of Tabular Integration by Parts (TIBP):

Example 1. Evaluate the $\int x^5 \cos x dx$

Solution:

Applying the LIATHE Rule, let $u = x^5$ and $dv = \cos x dx$. Using the algorithm of the TIBP, Table 2 can be formed as shown:

Table 2
TIBP for evaluating $\int x^5 \cos x dx$

S	D	I
+ →	x^5	$\cos x$
− →	$5x^4$	$\sin x$
+ →	$20x^3$	$-\cos x$
− →	$60x^2$	$-\sin x$
+ →	$120x$	$\cos x$
− →	120	$\sin x$
+ →	0	$-\cos x$

From Table 2, the resulting integral is $\int x^5 \cos x dx = x^5 \sin x + 5x^4 \cos x$

$$- 20x^3 \sin x - 60x^2 \cos x + 120x \sin x + 120 \cos x + C$$

Example 2. Evaluate the $\int x^4 \sinh x dx$.

Solution:

Applying the LIATHE Rule, let $u = x^4$ and $dv = \sinh x dx$ Using the algorithm of the TIBP, Table 3 can be formed.

Table 3
TIBP for evaluating $\int x^4 \sinh x dx$

S	D	I
+ →	x^4	$\sinh x$
- →	$4x^3$	$\cosh x$
+ →	$12x^2$	$\sinh x$
- →	$24x$	$\cosh x$
+ →	24	$\sinh x$
- →	0	$\cosh x$

The resulting integral is expressed as

$$\int x^4 \sinh x dx = x^4 \cosh x - 4x^3 \sinh x + 12x^2 \cosh x - 24x \sinh x + 24 \cosh x + C$$

Example 3. Evaluate the $\int x^3 e^x dx$.

Solution:

Applying the LIATHE Rule, let $u = x^3$ and $dv = e^x dx$. Using the algorithm of the TIBP, Table 4 can be formed.

Table 4
TIBP for evaluating $\int x^3 e^x dx$

S	D	I
+ →	x^3	e^x
- →	$3x^2$	e^x
+ →	$6x$	e^x
- →	6	e^x
+ →	0	e^x

Based on Table 4, the resulting integral is

$$\int x^3 e^x dx = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C$$

General forms of composite functions

The general forms of composite functions where Tabular Integration by Parts

(TIBP) can be applied are: $\int f(x^{1/n})dx$ and $\int f(\ln x)dx$.

Consider the integral of the form $\int f(x^{1/n})dx$. Using the substitution rule, let $z = x^{1/n}$, then $z^n = x$ and taking the derivative of both sides gives $nz^{n-1}dz = dx$. The original form can now be written as $\int f(z) \cdot nz^{n-1}dz$, which is an integral of a product of a polynomial function and a function f(z).

A similar approach can be done to the integral of the form $\int f(\ln x)dx$. Using the substitution rule, let $z = \ln x$, then $e^z = x$ and taking the derivative of both sides gives $e^z dz = dx$. The original form can now be written as $\int f(z) \cdot e^z dz$, which is an integral of a product of an exponential function and a function f(z).

Integrals of Composite Functions using TIBP

In evaluating the integrals of composite functions using the algorithm of TIBP, the general rule is that the inner function of the composition should have an inverse that is easy to evaluate. This technique can be used to evaluate the following examples illustrating the first form:

Example 4. Evaluate the $\int \sin(x^{1/2})dx$.

Solution:

Setting $z = x^{1/2}$ then $z^2 = x$ and $2zdz = dx$, the original integral can be written as $\int (\sin z)(2z)dz$. Applying LIATHE rule, let $u = 2z$ and $dv = \sin z dz$. Using the algorithm of the TIBP, Table 5 can be formed.

Table 5.

TIBP for evaluating $\int \sin(x^{1/2})dx$

S	D	I
+	$2z$	$\sin z$
-	2	$-\cos z$
+	0	$-\sin z$

From the table, the resulting integral is

$$\int (\sin z)(2z)dz = (2z)(-\cos z) - (2)(-\sin z) + C$$

Finally, expressing the result in terms of the original variable yields

$$\int \sin(x^{1/2})dx = -2(x^{1/2})\cos(x^{1/2}) + 2\sin(x^{1/2}) + C$$

Similarly, the previous technique can also be applied to $\int \cos(x^{1/2})dx$ which gives a resulting integral of

$$\int \cos(x^{1/2})dx = 2(x^{1/2})\sin(x^{1/2}) + 2\cos(x^{1/2}) + C$$

Example 5. Evaluate the $\int \cosh(x^{1/3})dx$.

Solution:

Setting $z = x^{1/3}$ then $z^3 = x$ and $3z^2dz = dx$, the original integral can be written as $\int (\cosh z)(3z^2)dz$. Applying LIATHE rule, let $u = 3z^2$ and $dv = \cosh z dz$. Using the algorithm of the TIBP, you can form Table 6.

Table 6

TIBP for evaluating $\int \cosh(x^{1/3})dx$

S	D	I
+	$3z^2$	$\cosh z$
-	$6z$	$\sinh z$
+	6	$\cosh z$
-	0	$\sinh z$

From Table 6, the resulting integral is

$$\int (\cosh z)(3z^2)dz = (3z^2)(\sinh z) - (6z)(\cosh z) + (6)(\sinh z) + C$$

Expressing the result in terms of the original variable gives

$$\int \cosh(x^{1/3})dx = (3x^{2/3})(\sinh x^{1/3}) - 6(x^{1/3})(\cosh x^{1/3}) + (6)(\sinh x^{1/3}) + C$$

Similarly, when the technique is applied to $\int \sinh(x^{1/3})dx$ gives

$$\int \sinh(x^{1/3})dx = (3x^{2/3})(\cosh x^{1/3}) - 6(x^{1/3})(\sinh x^{1/3}) + (6)(\cosh x^{1/3}) + C$$

Example 6. Evaluate the $\int e^{\sqrt{x}}dx$.

Solution:

Setting $z = x^{1/2}$ then $z^2 = x$ and $2zdz = dx$, the original integral can be written as $\int (e^z)(2z)dz$. Applying LIATHE rule, let $u = 2z$ and $dv = e^z dz$ .Using the algorithm of the TIBP, Table 7 can be formed.

Table 7

TIBP for evaluating $\int e^{\sqrt{x}}dx$

S	D	I
+	$2z$	e^z
-	2	e^z
+	0	e^z

From the table, the resulting integral is

$$\int (e^z)(2z)dz = (2z)(e^z) - (2)(e^z) + C$$

Expressing the result in terms of the original variable gives

$$\int e^{\sqrt{x}}dx = 2(x^{1/2})e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

Similarly, this technique can be used to evaluate the following examples illustrating the second form:

Example 7. Evaluate the $\int (\ln x)^2 dx$.

Solution:

Setting $z = \ln x$ then $e^z = x$ and $e^z dz = dx$, the original integral can be written as $\int (z)^2 e^z dz$. Applying LIATHE rule, let $u = z^2$ and $dv = e^z dz$. Using the algorithm of the TIBP, Table 8 can be formed. The resulting integral can be written as

$$\int (z)^2 e^z dz = z^2 e^z - 2ze^z + 2e^z + C$$

Expressing the result in terms of the original variable and simplifying gives

$$\int (\ln x)^2 dx = (\ln x)^2 e^{\ln x} - 2(\ln x)e^{\ln x} + 2e^{\ln x} + C$$

$$\int (\ln x)^2 dx = x(\ln x)^2 - 2x(\ln x) + 2x + C$$

Table 8

TIBP for evaluating $\int (\ln x)^2 dx$

S	D	I
+	z^2	e^z
-	$2z$	e^z
+	2	e^z
-	0	e^z

Example 8. Evaluate the $\int \sin(\ln x)dx$.

Solution:

Setting $z = \ln x$ then $e^z = x$ and $e^z dz = dx$, the original integral can be written as $\int(\sin z)(e^z)dz$. Applying the LIATHE rule, let $u=\sin z$ and $dv = e^z dz$. Using the algorithm of the TIBP, form the next table.

Table 9

TIBP for evaluating $\int e^z \sin z dz$

S	D	I
+	$\sin z$	e^z
-	$\cos z$	e^z
+	$-\sin z$	e^z

From Table 9, the resulting integral is

$$\int e^z \sin z dz = (\sin z)(e^z) - (\cos z)(e^z) + $$
$$+\int(-\sin z)(e^z)dz$$

This can be regarded as an equation to be solved for the unknown integral. Adding $\int e^z \sin z dz$ to both sides yields

$$2\int e^z \sin z dz = (\sin z)(e^z) - (\cos z)(e^z)$$

Dividing by 2, simplifying, and adding the constant of integration gives a result of

$$\int e^z \sin z dz = \frac{1}{2}e^z \sin z - \frac{1}{2}e^z \cos z + C$$

Expressing the result in terms of the original variable and simplifying gives

$$\int \sin(\ln x)dx = \frac{1}{2}e^{\ln x}\sin(\ln x) - \frac{1}{2}e^{\ln x}\cos(\ln x) + C$$

$$\int \sin(\ln x)dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x) + C$$

Conclusions

In light of the results of the study, the following conclusions were drawn:

1. The algorithm of the Tabular Integration by Parts (TIBP) can be used to evaluate the integrals of the product of polynomial functions and other elementary functions such as trigonometric, hyperbolic, and exponential functions.

2. The general forms of integrals of composite functions where TIBP can be used are $\int f(x^{1/n})dx$ and $\int f(\ln x)dx$.

3. In evaluating the integrals of composite functions using the TIBP, the inner function of the composition should have an inverse that is easy to evaluate.

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